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Propagation of internal Alfvén–acoustic–gravity waves in a perfectly conducting isothermal compressible fluid

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Internal Alfvén–acoustic–gravity waves propagating in an isothermal, perfectly electrically conducting, plane stratified, inviscid, compressible atmosphere permeated by a horizontal stratified magnetic field in which the mean horizontal velocity $U(z)$ depends on the height z only exhibit singular properties at the Doppler-shifted frequencies

$$\Omega_d = 0, \quad \pm\Omega_A, \quad \pm\Omega_A/(1+M^2)^{\frac{1}{2}}, \quad \pm(\Omega_c/2^{\frac{1}{2}})\,[1+M^2 \pm\{(1+M^2)^2 - 4\Omega_A^2/\Omega_c^2\}^{\frac{1}{2}}]^{\frac{1}{2}},$$

where Ω_A is the Alfvén frequency, Ω_c the sonic frequency and M the magnetic Mach number. The phenomenon of critical-layer absorption is studied using the momentum-transport approach of Booker & Bretherton (1967), the wave-packet approach (which is a consequence of the WKBJ approximation) of Bretherton (1966) and the technique involving wave normal curves of McKenzie (1973). The absorption effects are also illustrated, following Acheson (1972), by drawing ray trajectories. We find that the waves are absorbed at the critical levels $\Omega_d = \pm\Omega_A$ and $\pm\Omega_A/(1+M^2)^{\frac{1}{2}}$, and in particular we observe that these levels do not act like valves as observed by Acheson (1972). We also conclude that the combined effect of velocity shear and density and magnetic-field stratification is to increase the number of absorption levels.

1. Introduction

It has been pointed out (Acheson 1972; Rudraiah & Venkatachalappa 1972*a, b, c*, hereafter referred to as RV*a, b, c*) that hydromagnetic wave groups in rotating and non-rotating Boussinesq fluids, like hydrodynamic wave groups (Bretherton 1966, 1969; Booker & Bretherton 1967; Jones 1967), can exhibit critical-level behaviour. More recently McKenzie (1973) has discussed the general nature of critical levels for any type of wave propagation in a stratified medium and has shown that a critical level at which a wave packet is neither reflected nor transmitted can exist only if the wave normal curve, which is formed by taking the cross-section through the wave normal surface in the plane of propagation, possesses an asymptote which is parallel to the direction of variation of the properties of the medium through which the wave propagates.

The work related to critical-level behaviour of waves in a stratified conducting fluid in RV$a, b, c$ is mainly concerned with situations in which the speed of fluid flow is much less than that of sound in the medium and accelerations are slow compared with those associated with sound waves. Another important assumption made is the Boussinesq approximation, which amounts to the neglect of density variation except in the buoyancy term. This assumption that fluctuations in density occur principally as a result of thermal, rather than pressure, variations is a natural approximation in the case of a liquid, but is much more restrictive in the case of a compressible fluid. Conditions under which internal Alfvén–acoustic–gravity waves are important in geophysics and astrophysics are usually far removed from the idealization of a Boussinesq fluid in which the speed of fluid flow is much less than that of sound in the medium. In the meteorological case, variations in density and pressure within the troposphere can scarcely be regarded as small. In such circumstances we must study the propagation of waves in a compressible fluid without the Boussinesq approximation. Another important feature is that many aspects of upper-atmosphere dynamics and ionospheric irregularities may be explained in terms of atmospheric gravity waves (see Hines 1960, 1963, 1964, 1968). These waves and their interactions with ionization and magnetic field have many subtleties and hidden characteristics that must be explored.

Therefore, in the present study attention is focused on the effect of compressibility on hydromagnetic internal gravity waves. In particular, we study the propagation in the presence of an aligned magnetic field of Alfvén–acoustic–gravity waves in a compressible, stratified, inviscid, perfectly conducting, isothermal atmosphere in which the mean horizontal velocity in the x direction varies with the height z only. We consider a basic aligned magnetic field whose magnitude varies with height in such a manner as to render the Alfvén velocity constant for the entire atmosphere. It may be remarked that in the real atmosphere the density, pressure and magnetic field do change with height though not necessarily in the manner implied above. The assumptions of constant Alfvén velocity and constant temperature are made for mathematical simplicity so as to evolve the simplest model of a hydromagnetic atmosphere, and it is hoped that the physics of the problem are not materially changed.

The mathematical formulation and the corresponding solutions are discussed in §§2 and 3. In §4.1 we show, as in the case of incompressible fluid (RV 1972c), that the total momentum flux, which is the algebraic sum of the wave momentum fluxes in the fields and the material media, is conserved everywhere in the fluid except across the critical levels and discuss the attenuation of waves, using the transfer of momentum. Also, we discuss in §4.2, using wave-packet analysis, the mechanism of absorption, reflexion and transmission of waves at the critical levels and illustrate this, following Acheson (1973), through ray trajectories. These effects are also verified, in §4.3, by drawing asymptotes to the wave normal curve.

2. Equations of the problem

We consider a system of Cartesian axes with the z axis vertical. We assume the fluid to be compressible and ideal (zero viscosity, zero thermal conductivity and zero magnetic viscosity) with vertical density stratification. We consider an isothermal atmosphere, which gives a relatively simple approach to the physics of the problem and is sufficient for present purposes, although such an approach must limit the ultimate applicability and accuracy of the results. Under these assumptions the basic magnetohydrodynamic equations are

$$\rho_1\left[\frac{\partial \mathbf{q}}{\partial t}+(\mathbf{q}\,.\,\nabla)\,\mathbf{q}\right] = -\nabla p_1+\rho_1\mathbf{g}+\mu(\nabla\times\mathbf{H})\times\mathbf{H}, \tag{2.1}$$

$$\partial\rho_1/\partial t+\nabla\,.\,(\rho_1\mathbf{q}) = 0, \tag{2.2}$$

$$\partial p_1/\partial t+(\mathbf{q}\,.\,\nabla)\,p_1 = c^2[\partial\rho_1/\partial t+(\mathbf{q}\,.\,\nabla)\,\rho_1], \tag{2.3}$$

$$\partial\mathbf{H}/\partial t = \nabla\times(\mathbf{q}\times\mathbf{H}), \quad \nabla.\mathbf{H} = 0, \qquad (2.4), (2.5)$$

where $\mathbf{q}$ denotes the flow velocity, ρ_1 the density, p_1 the hydrodynamic pressure, c the (constant) sound speed, $\mathbf{H}$ the magnetic field and μ the magnetic permeability.

2.1. *Equilibrium configuration*

The compressible ideal fluid is assumed to have an x velocity component U varying in the vertical (z) direction. The material density $\rho_0(z)$ and the magnetic field $H_0(z)$, in the x direction, are assumed to be of the form

$$\rho_0(z) = \rho_c\exp(-\beta z), \quad H_0(z) = H_c\exp(-\tfrac{1}{2}\beta z), \tag{2.6}$$

where β is the reciprocal of the scale height and is written as

$$\beta = g(c^2/\gamma+\tfrac{1}{2}A^2)^{-1} = g\gamma\{c^2(1+\tfrac{1}{2}\gamma M^2)\}^{-1}, \tag{2.7}$$

where A denotes the Alfvén speed $(\mu H_0^2/\rho_0)^{\frac{1}{2}}$, $M = A/c$ is the magnetic Mach number and γ is the usual ratio of specific heats. For magnetostatic balance we have

$$dp_0/dz = -(g\rho_0+\mu H_0\,dH_0/dz), \tag{2.8}$$

where p_0 denotes the steady-state hydrodynamic pressure.

2.2. *Perturbed state*

On the equilibrium configuration discussed above we superimpose a small disturbance of the form

$$(U+u, v, w), \quad \rho_0+\rho, \quad p_0+p, \quad (H_0+h_x, h_y, h_z).$$

We assume that the disturbances are small enough compared with the basic state that higher-order terms in perturbed quantities can be neglected. Equations (2.1)–(2.5) then reduce to a set of linear partial differential equations which admits plane-wave solutions in which all perturbation quantities f may be written as

$$f = \mathrm{Re}\,[\hat{f}(z)\exp i(kx+ly-\sigma t)]. \tag{2.9}$$

Elimination of all dependent variables but w leads to the wave equation

$$\frac{d^2\hat{w}}{dz^2}-\left[\beta-\frac{2Qk\,dU/dz}{\Omega_d(\Omega_d^2-\Omega_A^2)(Q-\Omega_d^4)}\left(\Omega_A^2+\frac{P\Omega_d^4}{Q^2}\right)\right]\frac{d\hat{w}}{dz}$$
$$+\left[\frac{-\alpha^2\{(\Omega_d^2-\Omega_A^2)(Q+\Omega_c^2N'^2)+l^2k^{-2}N^2\Omega_A^2\Omega_d^2\}}{(\Omega_d^2-\Omega_A^2)(Q-\Omega_d^4)}\right.$$
$$+\frac{\beta k\,dU/dz}{\Omega_d}-\frac{2g\alpha^2\Omega_d k(dU/dz)P}{Q(\Omega_d^2-\Omega_A^2)(Q-\Omega_d^4)}$$
$$\left.-\frac{2Qk^2(dU/dz)^2}{(\Omega_d^2-\Omega_A^2)(Q-\Omega_d^4)}\left(\frac{\Omega_A^2}{\Omega_d^2}+\frac{P\Omega_d^2}{Q^2}\right)-\frac{k\,d^2U/dz^2}{\Omega_d}\right]\hat{w}=0, \quad (2.10)$$

where

$$P=(\Omega_d^2-\Omega_A^2)^2+l^2k^{-2}\Omega_d^4\Omega_A^2, \quad N^2=g\beta,$$
$$Q=(\Omega_d^2-\Omega_A^2)(\Omega_d^2-\Omega_c^2)-l^2k^{-2}\Omega_d^2\Omega_A^2,$$
$$\Omega_d=kU-\sigma,\quad \Omega_c=\alpha c,\quad \Omega_A=kA,\quad \alpha^2=k^2+l^2,\quad N'^2=g\beta-g^2/c^2.$$

The other perturbation quantities are related to w by the formulae

$$u=\frac{ik(\Omega_d^2-\Omega_A^2)}{Q}\left[-c^2\frac{dw}{dz}+\left(g+\frac{kc^2}{\Omega d}\frac{dU}{dz}\right)w\right]-\left[\frac{igl^2k^{-1}\Omega_A^2}{Q}-\frac{i\,dU/dz}{\Omega d}\right]w, \quad (2.11)$$

$$v=\frac{1}{\Omega_d^2-\alpha^2k^{-2}\Omega_A^2}\left[-\frac{il}{k^2}\Omega_A^2\left(\frac{dw}{dz}-\frac{k\,dU/dz}{\Omega d}w\right)+ilgw-ilc^2\nabla\cdot\mathbf{q}\right], \quad (2.12)$$

$$\nabla\cdot\mathbf{q}=\frac{\Omega_d^2-\Omega_A^2}{Q}\left[\Omega_d^2\frac{dw}{dz}-g\alpha^2w-k\Omega_d\frac{dU}{dz}w\right], \quad (2.13)$$

$$p=-(i/\Omega_d)[(\rho_0g+\mu H_0\,dH_0/dz)w-c^2\rho_0\nabla\cdot\mathbf{q}], \quad (2.14)$$

$$h_x=\frac{1}{\Omega_d}\left[-lH_0v+i\frac{d(H_0w)}{dz}-\frac{i}{\Omega_d}kH_0\frac{dU}{dz}w\right], \quad (2.15)$$

$$h_y=(kH_0/\Omega_d)v,\quad h_z=(kH_0/\Omega_d)w. \quad (2.16),(2.17)$$

When the basic flow is uniform the coefficients in (2.10) are constant and the equation has solution $w \propto \exp(imz)$, where m is a constant vertical wavenumber. However, when the basic flow is not uniform, the most striking feature of (2.10) is its singularities at

$$\Omega_d=0,\ \pm\Omega_A,\ \pm\frac{\Omega A}{(1+M^2)^{\frac{1}{2}}},\ \pm\left(\frac{\Omega_c}{2^{\frac{1}{2}}}\right)\left[1+M^2\pm\left\{(1+M^2)^2-\frac{4\Omega_A^2}{\Omega_c^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}. \quad (2.18)$$

In the case of two-dimensional disturbances (i.e. $l=0$), though, the governing wave equation is singular only at

$$\Omega_d=0,\ \pm\Omega_c,\ \pm\Omega_A/(1+M^2)^{\frac{1}{2}} \quad (2.19)$$

and not at $\Omega_d=\pm\Omega_A$, contrary to the three-dimensional case. It is of interest to note, as in the case of singular solutions, that although these singularities do not tend directly to the incompressible case of RVc in the limit $M\to 0$, i.e. $c\to\infty$, equation (2.10) gives the singularities of incompressible flow in the limit $M\to 0$.

In the following sections we discuss the wave propagation in the neighbourhood of the singular levels and indicate some possible consequences.

3. Solutions of the wave equation

The singularities (2.18) of (2.10) may evidently be regarded as a consequence of the loss of higher-order derivatives owing to the neglect of dissipative effects. In order to eliminate these singularities, viscosity, magnetic viscosity and heat conduction must be considered and an eighth-order linear differential equation must then be investigated. Alternatively, following Miles' (1961) analysis in connexion with the critical level for internal gravity waves in a shear flow, they may also be regarded as a consequence of our restriction to a single sinusoidal component given by (2.9). Accordingly, by posing an initial-value problem and then determining its asymptotic solution as $t \to \infty$, we should, even in the absence of dissipative effects, be able to match the solutions on the two sides of the critical level. It has, however, proved possible to resolve the singularities by a simpler means by following Booker & Bretherton (1967). For a detailed mathematical analysis and the physical significance of this method we may refer to Acheson (1972). We also note that it is possible to reveal the significance of these singularities by using the WKB method and following the analysis of Bretherton (1966).

In this section we follow Booker & Bretherton's (1967) analysis to resolve the singularities of (2.10). For this the power-series solutions of the wave equation (2.10) near the singular levels are obtained using the method of Frobenius and assuming that the velocity shear dU/dz is independent of height. Near the critical levels $\Omega_d = 0$, Ω_A the complete solutions of (2.10) are respectively of the forms

$$\hat{w} = A_1(z-z_0)^2\,[1+a_1(z-z_0)+a_2(z-z_0)^2+\ldots] + B_1(z-z_0)\,[1+b_1(z-z_0)+b_2(z-z_0)^2+\ldots] \quad (3.1)$$

and $$\hat{w} = [A_2+B_2\log(z-z_1)]\,[1+c_1(z-z_1)+\ldots]+B_2\sum_{i=0}^{\infty}\left[\frac{\partial c_i}{\partial \gamma}\right]_{\gamma=0}(z-z_1)^i, \quad (3.2)$$

where z_0 and z_1 are such that $\Omega_d = 0$ at $z = z_0$ and $\Omega_d = \Omega_A$ at $z = z_1$, A_1, A_2, B_1 and B_2 are constants of integration while $a_1, a_2, \ldots$, $b_1, b_2, \ldots$, and $c_1, c_2, \ldots$, are known constants. A similar solution can be obtained near the critical level $\Omega_d = -\Omega_A$. We note that solutions (3.1) and (3.2) are of the same form as those obtained by RVc near these critical levels in the case of incompressible conducting fluid. In other words, the behaviour of waves near these critical levels is the same in both incompressible and compressible conducting fluids. These waves are compared with the hydrodynamic incompressible solutions of Booker & Bretherton (1967) and the hydrodynamic compressible solutions of Hines (1968) at the end of this section.

The solution near the critical level $\Omega_d = \Omega_A/(1+M^2)^{\frac{1}{2}}$ is

$$\hat{w} = [A_3+B_3\log(z-z_2)]\,[1+e_1(z_1-z_2)+\ldots]+B_3\sum_{k=0}^{\infty}\left[\frac{\partial e_k}{\partial \gamma}\right]_{\gamma=0}(z-z_2)^k, \quad (3.3)$$

where z_2 is such that $\Omega_d = \Omega_A/(1+M^2)^{\frac{1}{2}}$ at $z = z_2$ and A_3 and B_3 are constants of integration. Note that the solution (3.3) is similar to the solution (3.2). In other words, the behaviour of waves near $\Omega_d = \Omega_A$ is the same as that of waves near $\Omega_d = \Omega_A/(1+M^2)^{\frac{1}{2}}$. The solution near the critical level

$$\Omega_d = (\Omega_c/2^{\frac{1}{2}})\,[1+M^2+\{(1+M^2)^2-4\Omega_A^2/\Omega_c^2\}^{\frac{1}{2}}]^{\frac{1}{2}},$$

i.e. near $z = z_3$, is

$$\hat{w} = A_4(z-z_3)^2\,[1+\bar{a}_1(z-z_3)+\ldots]+B_4[1+\bar{b}_1(z-z_3)+\ldots], \tag{3.4}$$

where A_4 and B_4 are constants of integration. Similar solutions can be obtained near the remaining critical levels.

It is of interest to know the behaviour of waves away from all these critical levels. For this, we restrict the value of Ω_d by

$$N'^2 \gg |Q|/\Omega_c^2, \tag{3.5}$$

which implies that for internal gravity waves the Brunt–Väisälä frequency N' should be much greater than all other frequencies. In the limit $c \rightarrow \infty$ this approximation becomes

$$N^2 \gg \Omega_d^2-\Omega_A^2, \tag{3.6}$$

which is the same as the one used by RVc in the case of incompressible conducting fluid.

Using the transformations

$$\hat{w} = \tilde{w}(z)\exp(\tfrac{1}{2}\beta z), \quad \zeta = (z-z_0)^{-1} \tag{3.7}$$

and the approximation (3.5), (2.10) becomes

$$\zeta^4\frac{d^2\tilde{w}}{d\zeta^2}+2\zeta^3\frac{d\tilde{w}}{d\zeta}-\frac{\alpha^2[(q^2/\zeta^2-\Omega_A^2)\,\Omega_c^2N'^2+l^2k^{-2}\Omega_A^2N^2q^2/\zeta^2]}{(q^2/\zeta^2-\Omega_A^2)\,(Q-q^4/\zeta^4)}\tilde{w} = 0, \tag{3.8}$$

where $\Omega_d = q/\zeta = k(dU/dz)/\zeta$. We note that (3.8) has similar properties to the corresponding Boussinesq equation of RVc. The solution of (3.8) around $\zeta = 0$ is

$$\tilde{w} = A_5(z-z_0)^{\frac{1}{2}+i\mu_m}\,[1+\bar{c}_1/(z-z_0)+\ldots]+B_5(z-z_0)^{\frac{1}{2}-i\mu_m}\,[1+\bar{d}_1/(z-z_0)+\ldots], \tag{3.9}$$

where

$$\mu_m = \left[\frac{1}{1+M^2}\left(J'+\frac{l^2}{k^2}M^2J\right)-\frac{1}{4}\right]^{\frac{1}{2}},$$

$$J' = \left(1+\frac{l^2}{k^2}\right)\frac{N'^2}{(dU/dz)^2}$$

is the modified Richardson number,

$$J = \frac{N^2}{(dU/dz)^2}$$

is the Richardson number and A_5 and B_5 are constants of integration. If we fix (see Booker & Bretherton 1967; Acheson 1972) the branch of the complex powers in (3.9) by taking

$$\left.\begin{aligned}&(z-z_0)^{\frac{1}{2}\pm i\mu_m} = |z-z_0|^{\frac{1}{2}}\exp\{\pm i\mu_m\log|z-z_0|\} \quad \text{for} \quad z > z_0,\\ &\text{it follows that}\\ &(z-z_0)^{\frac{1}{2}\pm i\mu_m} = -i\exp(\pm\mu_m\pi)\,|z-z_0|^{\frac{1}{2}}\exp\{\pm i\mu_m\log|z-z_0|\} \quad \text{for} \quad z < z_0.\end{aligned}\right\} \tag{3.10}$$

Now we find that the amplitudes above and below the critical levels differ by a factor of $\exp(\mu_m\pi)$, the first term in (3.9) representing an upward-propagating wave and the second a downward-propagating wave, as is demonstrated in § 4 by equations (4.5) and (4.7). We note that the solution (3.9) and the attenuation factor μ_m tend asymptotically to the hydrodynamic results of Hines (1968) in the limit $\Omega_A \to 0$ (i.e. $M \to 0$) and to the incompressible results of RVc in the limit $c \to \infty$ with a suitable density stratification factor β.

It is of interest to know the behaviour of waves in (i) the case of no propagation in the y direction (i.e. $l = 0$) and (ii) the hydrodynamic case (i.e. $H = 0$).

(i) *No propagation in the y direction.* The governing wave equation for this case can be obtained from (2.10) by letting $l \to 0$. The singularities of this equation are at

$$\Omega_d = 0, \quad \pm\Omega_A/(1+M^2)^{\frac{1}{2}}, \quad \pm\Omega_c.$$

The solutions of the wave equation near these singularities are respectively similar to (3.1), (3.2) and (3.4). In other words, near these critical levels the behaviour with two-dimensional propagation is the same as that with three-dimensional propagation.

(ii) *Hydrodynamic case.* When the hydromagnetic effects are absent the singularities of the governing wave equation are at $\Omega_d = 0, \ \pm\Omega_c$. The solution near $\Omega_d = 0$ is the same as the solution obtained by Hines (1968) in the case of two-dimensional motion with the modified ν defined by

$$\nu = \tfrac{1}{2} + i(J' - \tfrac{1}{4})^{\frac{1}{2}},$$

where

$$J' = \frac{N'^2(1+l^2/k^2)}{(dU/dz)^2}$$

is the modified Richardson number. Thus in hydrodynamics the attenuation increases in the case of three-dimensional propagation. The solution near $\Omega_d = \Omega_c$ is similar to (3.4).

To find the effect of magnetic field on critical levels it is of interest to compare the hydromagnetic waves given by (3.1) with the corresponding incompressible solutions of Booker & Bretherton (1967) and compressible solutions of Hines (1968). They showed that the waves are attenuated near the critical level $\Omega_d = 0$. However hydromagnetic waves, discussed in § 4, pass through the critical level $\Omega_d = 0$ without any attenuation and instead are absorbed at the critical levels $\Omega_d = \pm\Omega_A, \ \pm\Omega_A/(1+M^2)^{\frac{1}{2}}$. Therefore the behaviour of hydromagnetic waves near $\Omega_d = 0$ is entirely different from that of hydrodynamic waves at this critical level.

4. Critical-layer absorption

In this section we study the absorption of waves near the critical levels using momentum transfer to the mean flow, the group-velocity approach and the wave normal technique.

4.1. *Transfer of momentum to the mean flow*

The vertical fluxes in the x and y directions of horizontal momentum, which is the algebraic sum of the wave momentum fluxes in the fields and material media, are respectively given by

$$\left.\begin{aligned} M_x &= \rho_0\overline{uw} - \mu\overline{h_x h_z} = \tfrac{1}{2}kG, \\ M_y &= \rho_0\overline{vw} - \mu\overline{h_y h_z} = \tfrac{1}{2}lG, \end{aligned}\right\} \qquad (4.1)$$

where an overbar denotes an average over a horizontal wavelength,

$$G = \mathrm{Re}\left\{\rho_0 i \frac{\Omega_d^2-\Omega_A^2}{\Omega_d^2-\alpha^2 A^2}\left[\frac{c^2(\Omega_d^2-\Omega_A^2)}{Q}+\frac{A^2}{\Omega_d^2}\right]\frac{dw}{dz}w^*\right\} \tag{4.2}$$

and w^* is the complex conjugate of w. By differentiating (4.2) with respect to z and using (2.10) it may be shown that

$$dG/dz = 0. \tag{4.3}$$

Thus the mean flux of vertical momentum is independent of height. This is true at any level except the critical levels, where the use of (2.10) is invalid. Since the upward transfer of momentum has zero divergence there can be no transfer of momentum to the mean flow, except possibly across the critical levels. Hence the momentum flux can be taken as a measure of the strength of the wave.

The upward transfer of wave energy per unit area at any level will be the mean rate of working of the total pressure forces (including magnetic pressure) on the fluid above, i.e. $\overline{p_T w}$, where

$$p_T = p+\mu H_0 h_x. \tag{4.4}$$

Using the expressions for p and h_x we may easily show that

$$\overline{p_T w} = -M_f\Omega_d/(k+l), \quad M_f = M_x+M_y. \tag{4.5}$$

Now we discuss the nature of the momentum and energy flux across each critical level and interpret the solutions in terms of wave motion. From (4.1), using (3.1) and (3.4), we find that the momentum flux is continuous across the critical levels

$$\Omega_d = 0, \quad \pm\Omega_c 2^{-\frac{1}{2}}[1+M^2\pm\{(1+M^2)^2-4\Omega_A^2/\Omega_c^2\}^{\frac{1}{2}}]^{\frac{1}{2}}.$$

That is, the strength of the wave remains the same as it passes through these critical levels. In other words, waves are completely transmitted across these critical levels without any attenuation. However, near the critical level $\Omega_d = \Omega_A/(1+M^2)^{\frac{1}{2}}$ the total momentum flux, given by

$$M_{x,y} = \begin{cases} \mathrm{Re}\,\dfrac{2i\rho_0(k,l)\,A_3^* B_3(dU/dz)\,(c^2+A^2)^{\frac{5}{2}}}{kc^5\Omega_A} & \text{for} \quad z > z_2, \\[2ex] \mathrm{Re}\,\dfrac{2i\rho_0(k,l)\,(A_3^* B_3-|B_3|^2 i\pi)\,(dU/dz)\,(c^2+A^2)^{\frac{5}{2}}}{kc^5\Omega_A} & \text{for} \quad z < z_2, \end{cases} \tag{4.6}$$

is essentially discontinuous and hence momentum is transferred to the mean flow in the vicinity of this critical level. Similar behaviour occurs near the critical levels $\Omega_d = -\Omega_A/(1+M^2)^{\frac{1}{2}}$, $\pm\Omega_A$. Away from these critical levels the momentum flux is given by

$$M_{x,y} = \begin{cases} +\dfrac{(k,l)\,(c^2+A^2)^{\frac{1}{2}}}{k\,dU/dz}(|A_5|^2-|B_5|^2) \quad \text{for} \quad z > z_0, \\[2ex] -\dfrac{(k,l)\,(c^2+A^2)^{\frac{1}{2}}}{k\,dU/dz}[|A_5|^2\exp(2\mu_m\pi)-|B_5|^2\exp(-2\mu_m\pi)] \quad \text{for} \quad z < z_0. \end{cases} \tag{4.7}$$

The magnitudes of each term in (4.7) at a given distance above and below the critical levels are not the same. However, for the first term in (3.9) the energy flux $\overline{p_T w}$ given by (4.5) is positive both above and below the critical levels, while for the second term it is negative above and below the critical levels. In other words, the first wave is associated with upward transfer of energy and the second with downward transfer of energy. Thus, from (4.7), we find that the waves are attenuated at the critical layers by a factor $\exp(-2\mu_m \pi)$. This process of critical-layer absorption depends only on the gross features of the flow and not on the details of the critical layers. We note that the above attenuation factor reduces to the non-MHD attenuation factor of Hines (1968) in the limit $\Omega_A \to 0$.

In the two-dimensional case we find that waves are attenuated across the critical levels $\Omega_d = \pm \Omega_A/(1+M^2)^{\frac{1}{2}}$ and in the hydrodynamic situation waves are attenuated only across the critical level $\Omega_d = 0$.

4.2. *Group velocity near critical levels*

In the previous subsection we discussed the propagation of waves near the critical levels using the concept of momentum transport. However, when the horizontal velocity $U(z)$ varies only slightly over distances of the order of a wavelength the concept of a wave group is extremely useful. This is a time-dependent train of waves of sufficient regularity for a local frequency, wave-number and amplitude to be everywhere approximately defined though these may vary with position and time. We shall focus attention on the propagation of these quantities rather than on the individual wave crests of which the train is composed. There is therefore a fundamental difference between this subsection and the previous one, where σ, k and l are constants. If the frequency σ and the wavenumbers k, l and m vary with position and time we can formally define the group velocity (see Acheson 1972) as

$$\mathbf{u}_g = (\partial\sigma/\partial k, \partial\sigma/\partial l, \partial\sigma/\partial m).$$

Using the transformation

$$\hat{w} = \psi \exp\left(\int_0^z \tfrac{1}{2}\beta\, dz\right), \tag{4.8}$$

equation (3.10) can be written in the form

$$\frac{d^2\psi}{dz^2} + R\frac{d\psi}{dz} + S\psi = 0, \tag{4.9}$$

where R and S are easily obtainable from (2.10) and (4.8). Now, if $U(z)$, N, Ω_A and c do not vary very much over a wavelength, an internal Alfvén–acoustic–gravity wave with horizontal wavenumbers k and l and vertical wavenumber m satisfies the dispersion relation

$$\begin{aligned}\Omega_d^6 - [(\alpha^2+m^2+\tfrac{1}{4}\beta^2)(c^2+A^2)+\Omega_A^2]\,\Omega_d^4 + [(\alpha^2+m^2+\tfrac{1}{4}\beta^2)(c^2+A^2)\,\Omega_A^2 \\ + c^2\Omega_A^2(\alpha^2+m^2+\tfrac{1}{4}\beta^2) + \alpha^2(c^2+A^2)\,N^2 - \Omega_A^2 N^2 - g^2\alpha^2]\,\Omega_d^2 \\ - c^2\Omega_A^2[(\alpha^2+m^2+\tfrac{1}{4}\beta^2)\,\Omega_A^2 + \alpha^2(N^2-g^2/c^2)] = 0.\end{aligned} \tag{4.10}$$

Converting this equation into a formula for m we find that

$$m^2 = \frac{\alpha^2\{(N^2+\Omega_A^2-\Omega_d^2)\,Q+(\Omega_d^2-\Omega_A^2)\,(g^2\alpha^2-N^2\Omega_d^2)\}}{(\Omega_d^2-\Omega_A^2)\,(Q-\Omega_d^4)}-\frac{\beta^2}{4}. \tag{4.11}$$

Thus, as the wave group approaches the critical level $z = z_1$, at which $\Omega_d^2 = \Omega_A^2$, the vertical wavenumber m increases indefinitely and is given asymptotically by

$$(\Omega_d^2-\Omega_A^2)\,m_1^2 = l^2N^2. \tag{4.12}$$

Now the vertical component $\partial\sigma/\partial m$ of the group velocity at any level can be computed from (4.10), and the asymptotic behaviour of $\partial\sigma/\partial m$ corresponding to (4.12) is given by

$$\frac{\partial\sigma}{\partial m} = \frac{(2k\,dU/dz)^{\frac{3}{2}}\,\Omega_A^{\frac{1}{2}}}{lN}\,(z-z_1)^{\frac{3}{2}}.$$

Thus when $|z-z_1|$ is small the height of the wave packet satisfies the equation

$$dz/dt = a(z-z_1)^{\frac{3}{2}}, \tag{4.13}$$

where a is a constant. This may be integrated to give

$$(z-z_1)^{\frac{1}{2}} = -2/(at+b), \tag{4.14}$$

where b is another constant, and the wave packet thus slows down in such a way that it does not reach the critical level $z = z_1$ in a finite time. It is therefore neither transmitted nor reflected. As the critical level is approached

$$\partial\sigma/\partial k \to U+A, \quad \partial\sigma/\partial l \to 0.$$

Thus the group is effectively captured in this neighbourhood and constrained thereafter to propagate along the mean flow.

Also, from (4.11), we see that as the wave group approaches the critical level $z = z_2$, at which $\Omega_d^2 = \Omega_A^2/(1+M^2)$, m increases indefinitely and when $z-z_2$ is small the height of the wave packet qualitatively satisfies (4.14). Thus the wave group is captured in the neighbourhood of $z = z_2$ also. It is observed that, in two-dimensional propagation, waves are effectively captured only at $\Omega_d^2 = \Omega_A^2/(1+M^2)$ whereas in the hydrodynamic case waves are captured at $\Omega_d = 0$.

Apart from these levels there are other levels at which waves cease to propagate vertically. These levels, called reflexion levels, are given by (4.11) with $m = 0$. If a particular packet propagates towards a level $z = z_r$ where $m = 0$, the vertical group velocity decreases as $|z-z_r|^{\frac{1}{2}}$. Thus the time taken for the wave group to reach z_r is finite and hence it is reflected there.

A sketch of the ray trajectories (projected in the x, z plane) is drawn in figure 1 to show the reflexion and critical-layer absorption of waves in a compressible shear flow. These trajectories are drawn on the basis of an investigation of the ray-path slope dz/dx (see Acheson 1973; McKenzie 1973) at various points like critical and reflexion levels. The path drawn is that appropriate to σ, k and l all positive. Thus a wave generated somewhere above or below its corresponding critical level and initially moving towards λ_γ (i.e. where $m = 0$) is reflected towards its critical level and captured there.

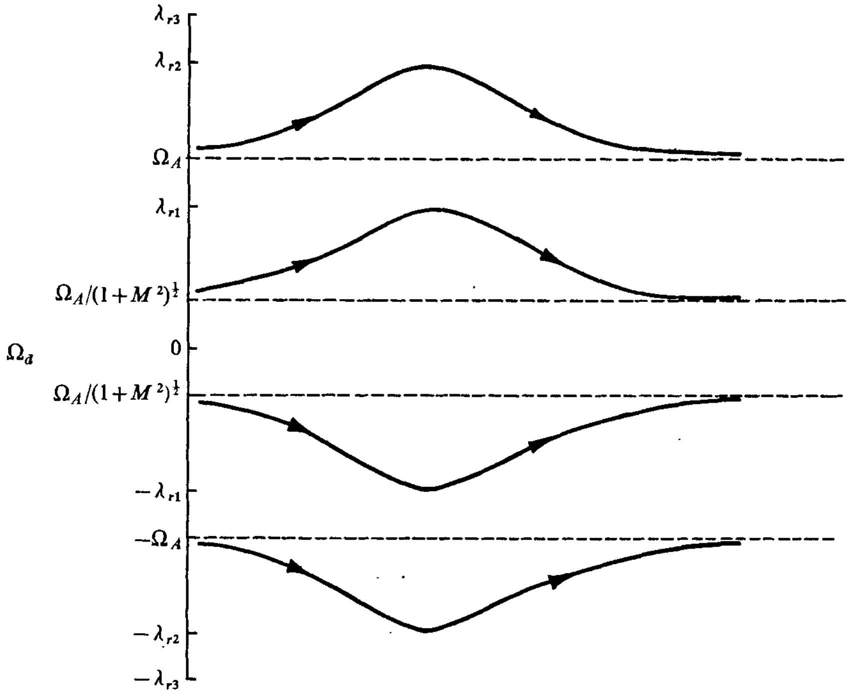

FIGURE 1. Ray trajectories illustrating the critical-layer absorption for Alfvén–acoustic–gravity waves in a shear flow. $k, l, \sigma > 0$.

4.3. *Wave normal curve*

These results also confirm the more general result (see McKensie 1973) obtained by tracing wave normal curves. If a wave normal curve, formed by taking the cross-section through the wave normal surface in the plane of propagation, possesses an asymptote, a critical level can exist provided that the properties of the medium vary in the direction parallel to the asymptote. It is clear from (4.11) and figure 2 that the wave normal curve appropriate to a slow magneto-acoustic gravity wave is asymptotic to the lines

$$k = \pm\,\Omega_d/A, \quad \pm\,\Omega_d(A^{-2}+c^{-2})^{\frac{1}{2}}. \tag{4.15}$$

Thus, since the properties of the medium (i.e. the Doppler-shifted frequency Ω_d) vary with z, a critical level can exist at $z = z_{1,2}$, where $z_{1,2}$ correspond to

$$k^2 = \Omega_d^2/A^2, \quad \Omega_d^2(A^{-2}+c^{-2}).$$

It is of interest to compare the wave normal curves (figure 2) in the presence of velocity shear and density stratification with those in the absence of these effects discussed by McKenzie (1973). Figure 2 reveals that (4.15) gives four absorption levels whereas two levels were predicted by McKenzie (1973). In other words the combined effect of velocity shear and density stratification on magneto-acoustic internal gravity waves is to increase the number of absorption levels.

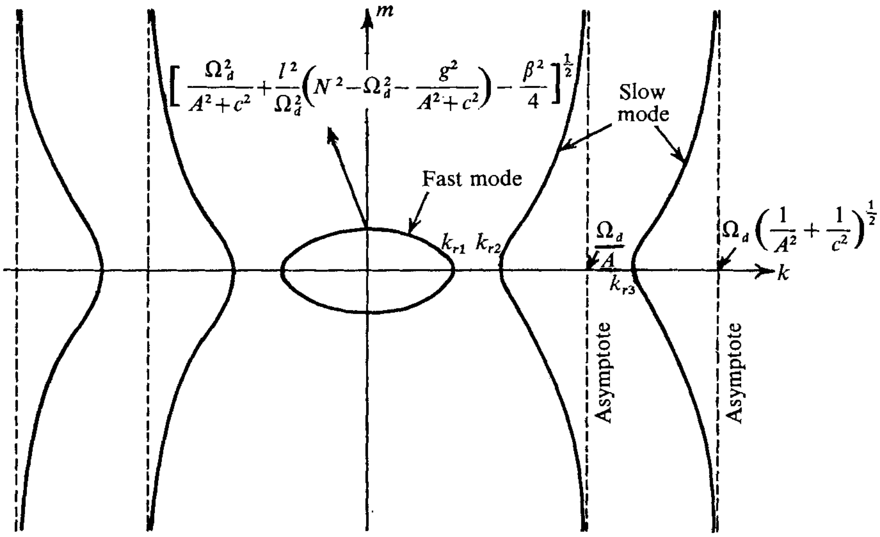

FIGURE 2. Wave normal curves for Alfvén–acoustic–gravity waves.

5. Conclusion

It has been shown that magneto-acoustic gravity waves exhibit singular behaviour at

$$\Omega_d = 0, \quad \pm\Omega_A, \quad \pm\frac{\Omega_A}{(1+M^2)^{\frac{1}{2}}}, \quad \pm\frac{\Omega_c}{2^{\frac{1}{2}}}\left[1+M^2 \pm\left\{(1+M^2)^2 - \frac{4\Omega_A^2}{\Omega_c^2}\right\}^{\frac{1}{2}}\right]^{\frac{1}{2}}$$

in contrast to the singularities at $\Omega_d = 0$, $\pm\Omega_c$ in the case of hydrodynamic compressible fluid discussed by Hines (1968) and $\Omega_d = 0$, $\pm\Omega_A$ in the case of perfectly conducting inviscid Boussinesq fluid investigated by RVc. In particular, two-dimensional magneto-acoustic gravity waves exhibit singular behaviour only at $\Omega_d = 0$, $\pm\Omega_A/(1+M^2)^{\frac{1}{2}}$. From this we conclude that the combined effect of magnetic field and compressibility is to increase the number of critical levels. The propagation of waves near these critical levels was discussed using (*a*) momentum transfer to the mean flow (Booker & Bretherton 1967), (*b*) the group-velocity approach (Bretherton 1966) and (*c*) the wave normal curve approach (McKenzie 1973). The first approach shows that internal gravity waves with frequency σ and horizontal wavenumber components k and l propagating in a compressible shear flow are highly attenuated, by a factor $\exp(-2\mu_m\pi)$, as they pass through the critical levels $\Omega_d = \pm\Omega_A$, $\pm\Omega_A/(1+M^2)^{\frac{1}{2}}$, but will pass through other critical levels without any attenuation. This behaviour is similar to that discussed by Hines (1968) in the case of hydrodynamic compressible shear flow, where he has proved that internal gravity waves are attenuated as they pass through the critical level $\Omega_d = 0$ but will pass through the other critical levels without any attenuation.

When this absorption mechanism is studied through the group-velocity approach (which is a consequence of WKBJ approximation), we find that a wave group travelling with the appropriate group velocity is neither transmitted nor reflected but is completely absorbed near the critical levels $\Omega_d = \pm\Omega_A$, $\pm\Omega_A/(1+M^2)^{\frac{1}{2}}$. Near these critical levels the vertical wavelength becomes very small and the motion is entirely in the horizontal direction. However, in the case of no propagation in the y direction (i.e. $l = 0$) the waves are completely absorbed near the critical levels $\Omega_d = \pm\Omega_A/(1+M^2)^{\frac{1}{2}}$.

This absorption phenomenon was also investigated by drawing the wave normal curves following McKenzie (1973). Comparison of the wave normal curves (figure 2), discussed in § 4.2, with those for magneto-acoustic waves in the absence of shear flow and density stratification (McKenzie 1973) reveals certain novel features. Magneto-acoustic waves with frequency σ and horizontal wavenumbers k and l in a fluid permeated by a shear magnetic field will be completely absorbed at each of the two critical levels

$$k = \pm\sigma(1+M^2)^{\frac{1}{2}}/A.$$

In contrast to this, magneto-acoustic internal gravity waves with frequency σ and horizontal wavenumbers k and l in a compressible shear flow $(U(z), 0, 0)$ with density stratification $e^{-\beta z}$ will be completely absorbed at each of the four critical levels

$$k = \pm\Omega_d/A, \quad \pm\Omega_d(1+M^2)^{\frac{1}{2}}/A.$$

These absorption effects were also illustrated by drawing ray trajectories (figure 1). From comparison of these results we conclude that the combined effect of velocity shear and density stratification is to increase the number of absorption levels.

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